

# Examples of certain kind of minimal orbits of Hermann actions

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## Abstract

We give examples of certain kind of minimal orbits of Hermann actions and discuss whether each of the examples is austere.

## 1 Introduction

Let  $N = G/K$  be a symmetric space of compact type equipped with the  $G$ -invariant metric induced from the Killing form of the Lie algebra of  $G$ . Let  $H$  be a symmetric subgroup of  $G$  (i.e.,  $(\text{Fix } \tau)_0 \subset H \subset \text{Fix } \tau$  for some involution  $\tau$  of  $G$ ), where  $\text{Fix } \tau$  is the fixed point group of  $\tau$  and  $(\text{Fix } \tau)_0$  is the identity component of  $\text{Fix } \tau$ . The natural action of  $H$  on  $N$  is called a *Hermann action* (see [HPTT], [Kol]). Let  $\theta$  be an involution of  $G$  with  $(\text{Fix } \theta)_0 \subset K \subset \text{Fix } \theta$ . According to [Co], when  $G$  is simple, we may assume that  $\theta \circ \tau = \tau \circ \theta$  by replacing  $H$  to a suitable conjugate group of  $H$  if necessary except for the following three Hermann action:

- (i)  $Sp(p+q) \curvearrowright SU(2p+2q)/S(U(2p-1) \times U(2q+1))$  ( $p \geq q+2$ ),
- (ii)  $U(p+q+1) \curvearrowright Spin(2p+2q+2)/Spin(2p+1) \times_{\mathbf{Z}_2} Spin(2q+1)$  ( $p \geq q+1$ ),
- (iii)  $Spin(3) \times_{\mathbf{Z}_2} Spin(5) \curvearrowright Spin(8)/\mu(Spin(3) \times_{\mathbf{Z}_2} Spin(5))$ ,

where  $\mu$  is the triality automorphism of  $Spin(8)$ . Here we note that we remove transitive Hermann actions.

**Assumption.** In the sequel, we assume that  $\theta \circ \tau = \tau \circ \theta$ . Then the Hermann action  $H \curvearrowright G/K$  is said to be *commutative*.

Let  $\mathfrak{g}, \mathfrak{k}$  and  $\mathfrak{h}$  be the Lie algebras of  $G, K$  and  $H$ , respectively. Denote the involutions of  $\mathfrak{g}$  induced from  $\theta$  and  $\tau$  by the same symbols  $\theta$  and  $\tau$ , respectively. Set  $\mathfrak{p} := \text{Ker}(\theta + \text{id})$  and  $\mathfrak{q} := \text{Ker}(\tau + \text{id})$ . The vector space  $\mathfrak{p}$  is identified with  $T_{eK}(G/K)$ , where  $e$  is the identity element of  $G$ . Denote by  $B_{\mathfrak{g}}$  the Killing form of  $\mathfrak{g}$ .

Give  $G/K$  the  $G$ -invariant metric arising from  $B_{\mathfrak{g}}|_{\mathfrak{p} \times \mathfrak{p}}$ . Take a maximal abelian subspace  $\mathfrak{b}$  of  $\mathfrak{p} \cap \mathfrak{q}$ . For each  $\beta \in \mathfrak{b}^*$ , we set  $\mathfrak{p}_\beta := \{X \in \mathfrak{p} \mid \text{ad}(b)^2(X) = -\beta(b)^2 X \ (\forall b \in \mathfrak{b})\}$  and  $\Delta' := \{\beta \in \mathfrak{b}^* \setminus \{0\} \mid \mathfrak{p}_\beta \neq \{0\}\}$ . This set  $\Delta'$  is a root system. Let  $\Pi' = \{\beta_1, \dots, \beta_r\}$  be the simple root system of the positive root system  $\Delta'_+$  of  $\Delta'$  under a lexicographic ordering of  $\mathfrak{b}^*$ . Set  $\Delta'_+{}^V := \{\beta \in \Delta'_+ \mid \mathfrak{p}_\beta \cap \mathfrak{q} \neq \{0\}\}$  and  $\Delta'_+{}^H := \{\beta \in \Delta'_+ \mid \mathfrak{p}_\beta \cap \mathfrak{h} \neq \{0\}\}$ . Define a subset  $\tilde{C}$  of  $\mathfrak{b}$  by

$$\tilde{C} := \{b \in \mathfrak{b} \mid 0 < \beta(b) < \pi \ (\forall \beta \in \Delta'_+{}^V), \ -\frac{\pi}{2} < \beta(b) < \frac{\pi}{2} \ (\forall \beta \in \Delta'_+{}^H)\}.$$

The closure  $\overline{\tilde{C}}$  of  $\tilde{C}$  is a simplicial complex. Set  $C := \text{Exp}(\tilde{C})$ , where  $\text{Exp}$  is the exponential map of  $G/K$  at  $eK$ . Each principal  $H$ -orbit passes through only one point of  $C$  and each singular  $H$ -orbit passes through only one point of  $\text{Exp}(\partial\tilde{C})$ . For each simplex  $\sigma$  of  $\tilde{C}$ , only one minimal  $H$ -orbit through  $\text{Exp}(\sigma)$  exists. See proofs of Theorems A and B in [K2] (also [I]) about this fact. For  $\beta \in \Delta'_+$ , we set  $\beta = \sum_{i=1}^r n_i^\beta \beta_i$ ,  $m_\beta := \dim \mathfrak{p}_\beta$ ,  $m_\beta^V := \dim(\mathfrak{p}_\beta \cap \mathfrak{q})$  and  $m_\beta^H := \dim(\mathfrak{p}_\beta \cap \mathfrak{h})$ . Let  $Z_0$  be a point of  $\mathfrak{b}$ . We consider the following two conditions for  $Z_0$ :

$$(I) \quad \left\{ \begin{array}{l} \beta(Z_0) \equiv 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6} \pmod{\pi} \ (\forall \beta \in \Delta'_+) \ \& \\ \sum_{\substack{\beta \in \Delta'_+{}^V \text{ s.t. } \beta(Z_0) \equiv \frac{\pi}{6} \pmod{\pi}}} 3n_i^\beta m_\beta^V + \sum_{\substack{\beta \in \Delta'_+{}^V \text{ s.t. } \beta(Z_0) \equiv \frac{\pi}{3} \pmod{\pi}}} n_i^\beta m_\beta^V \\ + \sum_{\substack{\beta \in \Delta'_+{}^H \text{ s.t. } \beta(Z_0) \equiv \frac{2\pi}{3} \pmod{\pi}}} 3n_i^\beta m_\beta^H + \sum_{\substack{\beta \in \Delta'_+{}^H \text{ s.t. } \beta(Z_0) \equiv \frac{5\pi}{6} \pmod{\pi}}} n_i^\beta m_\beta^H \\ = \sum_{\substack{\beta \in \Delta'_+{}^V \text{ s.t. } \beta(Z_0) \equiv \frac{2\pi}{3} \pmod{\pi}}} n_i^\beta m_\beta^V + \sum_{\substack{\beta \in \Delta'_+{}^V \text{ s.t. } \beta(Z_0) \equiv \frac{5\pi}{6} \pmod{\pi}}} 3n_i^\beta m_\beta^V \\ + \sum_{\substack{\beta \in \Delta'_+{}^H \text{ s.t. } \beta(Z_0) \equiv \frac{\pi}{6} \pmod{\pi}}} n_i^\beta m_\beta^H + \sum_{\substack{\beta \in \Delta'_+{}^H \text{ s.t. } \beta(Z_0) \equiv \frac{\pi}{3} \pmod{\pi}}} 3n_i^\beta m_\beta^H \\ (i = 1, \dots, r). \end{array} \right.$$

and

$$(II) \quad \left\{ \begin{array}{l} \beta(Z_0) \equiv 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4} \pmod{\pi} \ (\forall \beta \in \Delta'_+) \ \& \\ \sum_{\substack{\beta \in \Delta'_+{}^V \text{ s.t. } \beta(Z_0) \equiv \frac{\pi}{4} \pmod{\pi}}} n_i^\beta m_\beta^V + \sum_{\substack{\beta \in \Delta'_+{}^H \text{ s.t. } \beta(Z_0) \equiv \frac{3\pi}{4} \pmod{\pi}}} n_i^\beta m_\beta^H \\ = \sum_{\substack{\beta \in \Delta'_+{}^V \text{ s.t. } \beta(Z_0) \equiv \frac{3\pi}{4} \pmod{\pi}}} n_i^\beta m_\beta^V + \sum_{\substack{\beta \in \Delta'_+{}^H \text{ s.t. } \beta(Z_0) \equiv \frac{\pi}{4} \pmod{\pi}}} n_i^\beta m_\beta^H \\ (i = 1, \dots, r). \end{array} \right.$$

Denote by  $L$  the isotropy group of  $H$  at  $\text{Exp } Z_0$ . Denote by  $\mathfrak{h}$  (resp.  $\mathfrak{l}$ ) the Lie algebra of  $H$  (resp.  $L$ ) and  $B_{\mathfrak{g}}$  the Killing form of  $\mathfrak{g}$ . Also, denote by  $g_I$  the induced metric

on the submanifold  $M$  in  $G/K$  and  $\nabla^\perp$  the normal connection of the submanifold  $M$ . In the case where  $(\mathfrak{h}, \mathfrak{l})$  admits a reductive decomposition  $\mathfrak{h} = \mathfrak{l} + \mathfrak{m}$ , we denote the canonical connection of the principal  $L$ -bundle  $\pi : H \rightarrow H/L (= M)$  with respect to this reductive decomposition by  $\omega_{\mathfrak{m}}$ . Let  $F^\perp(M)$  be the normal frame bundle of  $M$ . Define a map  $\eta : H \rightarrow F^\perp(M)$  by  $\eta(h) = h_* u_0$  ( $h \in H$ ), where  $u_0$  is an arbitrary fixed element of  $F^\perp(M)_{\text{Exp } Z_0}$ , where  $F^\perp(M)_{\text{Exp } Z_0}$  is the fibre of  $F^\perp(M)$  over  $\text{Exp } Z_0$ . This map  $\eta$  is an embedding. By identifying  $H$  with  $\eta(H)$ , we regard  $\pi : H \rightarrow H/L (= M)$  as a subbundle of  $F^\perp(M)$ . Denote by the same symbol  $\omega_{\mathfrak{m}}$  the connection of  $F^\perp(M)$  induced from  $\omega_{\mathfrak{m}}$  and  $\nabla^{\omega_{\mathfrak{m}}}$  the linear connection on  $T^\perp M$  associated with  $\omega_{\mathfrak{m}}$ .

In this paper, we prove the following results for the orbit  $M = H(\text{Exp } Z_0)$  of the Hermann action  $H \curvearrowright G/K$ .

**Theorem A.** *If  $Z_0$  satisfies the condition (I) or (II), then the orbit  $M$  is a minimal submanifold satisfying the following conditions:*

- (i)  $(\mathfrak{h}, \mathfrak{l})$  admits a reductive decomposition  $\mathfrak{h} = \mathfrak{l} + \mathfrak{m}$  such that  $B_{\mathfrak{g}}(\mathfrak{l}, \mathfrak{m}) = 0$ ,
- (ii)  $\nabla^\perp = \nabla^{\omega_{\mathfrak{m}}}$  holds.

Also,  $\bigcap_{v \in T_x^\perp M} \text{Ker } A_v$  is equal to

$$\begin{aligned} g_{0*}(\mathfrak{z}_{\mathfrak{p} \cap \mathfrak{h}}(\mathfrak{b})) + & \sum_{\substack{\beta \in \Delta'_+{}^V \text{ s.t. } \beta(Z_0) \equiv \frac{\pi}{2} \pmod{\pi}}} g_{0*}(\mathfrak{p}_\beta \cap \mathfrak{q}) \\ + & \sum_{\substack{\beta \in \Delta'_+{}^H \text{ s.t. } \beta(Z_0) \equiv 0 \pmod{\pi}}} g_{0*}(\mathfrak{p}_\beta \cap \mathfrak{h}), \end{aligned}$$

where  $\mathfrak{z}_{\mathfrak{p} \cap \mathfrak{h}}(\mathfrak{b})$  is the centralizer of  $\mathfrak{b}$  in  $\mathfrak{p} \cap \mathfrak{h}$ .

Let  $M$  be a submanifold in a Riemannian manifold  $N$ . If, for any unit normal vector  $v$ , the spectrum of the shape operator  $A_v$  is invariant with respect to the  $(-1)$ -multiple (with considering the multiplicities), then  $M$  is called an *austere submanifold*. By using Theorem A, we can show the following fact.

**Theorem B.** *Assume that  $Z_0$  satisfies the condition (I) or (II). If  $m_\beta^V = m_\beta^H$  for all  $\beta \in \Delta'_+$  and if  $Z_0$  satisfies  $\beta(Z_0) \equiv 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4} \pmod{\pi}$  for all  $\beta \in \Delta'_+$ , then the orbit  $M$  is an austere submanifold satisfying the conditions (i) and (ii) in Theorem A.*

*Remark 1.1.* The austere orbits of the commutative Hermann actions were classified in [I].

Also, we can show the following facts.

**Theorem C.** Assume that  $Z_0$  satisfies the condition (I). In particular, if  $\Delta'_+{}^V \cap \Delta'_+{}^H = \emptyset$ , if  $\beta(Z_0) \equiv 0, \frac{\pi}{3}, \frac{2\pi}{3} \pmod{\pi}$  for all  $\beta \in \Delta'_+{}^V$  and if  $\beta(Z_0) \equiv \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6} \pmod{\pi}$  for all  $\beta \in \Delta'_+{}^H$ , then  $M$  is a minimal submanifold satisfying the conditions (i), (ii) in Theorem A. Furthermore, if the cohomogeneity of the  $H$ -action is equal to the rank of  $G/K$ , then  $(g_I)_{eL} = \frac{3}{4}B_{\mathfrak{g}}|_{\mathfrak{m} \times \mathfrak{m}}$  and  $\bigcap_{v \in T_x^\perp M} \text{Ker } A_v = \{0\}$  hold.

**Theorem D.** Assume that  $Z_0$  satisfies the condition (I). In particular, if  $\Delta'_+{}^V \cap \Delta'_+{}^H = \emptyset$ , if  $\beta(Z_0) \equiv 0, \frac{\pi}{6}, \frac{5\pi}{6} \pmod{\pi}$  for all  $\beta \in \Delta'_+{}^V$  and if  $\beta(Z_0) \equiv \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3} \pmod{\pi}$  for all  $\beta \in \Delta'_+{}^H$ , then  $M$  is a minimal submanifold satisfying the conditions (i), (ii) in Theorem A. Furthermore, if the cohomogeneity of the  $H$ -action is equal to the rank of  $G/K$ , then  $(g_I)_{eL} = \frac{1}{4}B_{\mathfrak{g}}|_{\mathfrak{m} \times \mathfrak{m}}$  and  $\bigcap_{v \in T_x^\perp M} \text{Ker } A_v = \{0\}$  hold.

**Theorem E.** Assume that  $Z_0$  satisfies the condition (II). In particular, if  $\Delta'_+{}^V \cap \Delta'_+{}^H = \emptyset$ , if  $\beta(Z_0) \equiv 0, \frac{\pi}{4}, \frac{3\pi}{4} \pmod{\pi}$  for all  $\beta \in \Delta'_+{}^V$  and if  $\beta(Z_0) \equiv \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4} \pmod{\pi}$  for all  $\beta \in \Delta'_+{}^H$ , then  $M$  is a minimal submanifold satisfying the conditions (i), (ii) in Theorem A. Furthermore, if the cohomogeneity of the  $H$ -action is equal to the rank of  $G/K$ , then  $(g_I)_{eL} = \frac{1}{2}B_{\mathfrak{g}}|_{\mathfrak{m} \times \mathfrak{m}}$  and  $\bigcap_{v \in T_x^\perp M} \text{Ker } A_v = \{0\}$  hold.

**Theorem F.** If  $\Delta'_+{}^V \cap \Delta'_+{}^H = \emptyset$ , if  $\beta(Z_0) \equiv 0, \frac{\pi}{2} \pmod{\pi}$  for all  $\beta \in \Delta'_+{}^V$ , then  $M$  is a totally geodesic submanifold satisfying the conditions (i), (ii) in Theorem A. Furthermore, if the cohomogeneity of the  $H$ -action is equal to the rank of  $G/K$ , then  $(g_I)_{eL} = B_{\mathfrak{g}}|_{\mathfrak{m} \times \mathfrak{m}}$  holds.

*Remark 1.2.* (i) If  $H = K$  then we have  $\Delta'_+{}^H = \emptyset$  and hence  $\Delta'_+{}^V \cap \Delta'_+{}^H = \emptyset$ .

(ii) In Theorems C~F, when  $G$  is simple, there exists an inner automorphism  $\rho$  of  $G$  with  $\rho(K) = H$  by Proposition 4.39 of [I].

In the final section, we give examples of Hermann actions  $H \curvearrowright G/K$  and  $Z_0 \in \mathfrak{b}$  as in Theorems B, C and F.

## 2 Basic notions and facts

In this section, we recall some basic notions and facts.

### Shape operators of orbits of Hermann actions

Let  $H \curvearrowright G/K$  be a Hermann action and  $\theta$  (resp.  $\tau$ ) an involution of  $G$  with  $(\text{Fix } \theta)_0 \subset K \subset \text{Fix } \theta$  (resp.  $(\text{Fix } \tau)_0 \subset H \subset \text{Fix } \tau$ ). Assume that  $\theta \circ \tau = \tau \circ \theta$ .

Let  $\mathfrak{k}, \mathfrak{p}, \mathfrak{h}, \mathfrak{q}, \mathfrak{b}, \mathfrak{p}_\beta, \Delta', \Delta'_+{}^V$  and  $\Delta'_+{}^H$  be as in Introduction. Fix  $Z_0 \in \mathfrak{b}$ . Set  $M := H(\text{Exp } Z_0)$  and  $g_0 := \exp Z_0$ , where  $\text{Exp}$  is the exponential map of  $G/K$  at  $eK$  and  $\exp$  is the exponential map of  $G$ . Set

$$\Delta'_{Z_0}{}^V := \{\beta \in \Delta'_+{}^V \mid \beta(Z_0) \equiv 0 \pmod{\pi}\}$$

and

$$\Delta'_{Z_0}{}^H := \{\beta \in \Delta'_+{}^H \mid \beta(Z_0) \equiv \frac{\pi}{2} \pmod{\pi}\}.$$

Denote by  $A$  the shape tensor of  $M$ . The tangent space  $T_{\text{Exp } Z_0} M$  of  $M$  at  $\text{Exp } Z_0$  is given by

$$(2.1) \quad T_{\text{Exp } Z_0} M = g_{0*} \left( \mathfrak{z}_{\mathfrak{p} \cap \mathfrak{h}}(\mathfrak{b}) + \sum_{\beta \in \Delta'_{Z_0}{}^V \setminus \Delta'_{Z_0}{}^V} (\mathfrak{p}_\beta \cap \mathfrak{q}) + \sum_{\beta \in \Delta'_{Z_0}{}^H \setminus \Delta'_{Z_0}{}^H} (\mathfrak{p}_\beta \cap \mathfrak{h}) \right)$$

and hence

$$(2.2) \quad T_{\text{Exp } Z_0}^\perp M = g_{0*} \left( \mathfrak{b} + \sum_{\beta \in \Delta'_{Z_0}{}^V} (\mathfrak{p}_\beta \cap \mathfrak{q}) + \sum_{\beta \in \Delta'_{Z_0}{}^H} (\mathfrak{p}_\beta \cap \mathfrak{h}) \right).$$

Denote by  $L$  the isotropy group of the  $H$ -action at  $\text{Exp } Z_0$ . The slice representation  $\rho_{Z_0}^S : L \rightarrow GL(T_{\text{Exp } Z_0}^\perp M)$  of the  $H$ -action at  $\text{Exp } Z_0$  is given by  $\rho_{Z_0}^S(h) = h_* \text{Exp } Z_0|_{T_{\text{Exp } Z_0}^\perp M}$  ( $h \in H_{Z_0}$ ). Then we have  $\bigcup_{h \in H_{Z_0}} \rho_{Z_0}^S(h)(g_{0*} \mathfrak{b}) = T_{\text{Exp } Z_0}^\perp M$  and

$$(2.3) \quad \begin{aligned} A_{\rho_{Z_0}^S(h)(g_{0*} v)}|_{\rho_{Z_0}^S(h)(g_{0*}(\mathfrak{z}_{\mathfrak{p} \cap \mathfrak{h}}(\mathfrak{b})))} &= 0, \\ A_{\rho_{Z_0}^S(h)(g_{0*} v)}|_{\rho_{Z_0}^S(h)(g_{0*}(\mathfrak{p}_\beta \cap \mathfrak{q}))} &= -\frac{\beta(v)}{\tan \beta(Z_0)} \text{id} \quad (\beta \in \Delta'_{Z_0}{}^V \setminus \Delta'_{Z_0}{}^V), \\ A_{\rho_{Z_0}^S(h)(g_{0*} v)}|_{\rho_{Z_0}^S(h)(g_{0*}(\mathfrak{p}_\beta \cap \mathfrak{h}))} &= \beta(v) \tan \beta(Z_0) \text{id} \quad (\beta \in \Delta'_{Z_0}{}^H \setminus \Delta'_{Z_0}{}^H), \end{aligned}$$

where  $h \in L$  and  $v \in \mathfrak{b}$ .

### The canonical connection

Let  $H/L$  be a reductive homogeneous space and  $\mathfrak{h} = \mathfrak{l} + \mathfrak{m}$  be a reductive decomposition (i.e.,  $[\mathfrak{l}, \mathfrak{m}] \subset \mathfrak{m}$ ), where  $\mathfrak{h}$  (resp.  $\mathfrak{l}$ ) is the Lie algebra of  $H$  (resp.  $L$ ). Also, let  $\pi : P \rightarrow H/L$  be a principal  $G$ -bundle, where  $G$  is a Lie group. Assume that  $H$  acts on  $P$  as  $\pi(h \cdot u) = h \cdot \pi(u)$  for any  $u \in P$  and any  $h \in H$ . Then there uniquely exists a connection  $\omega$  of  $P$  such that, for any  $X \in \mathfrak{m}$  and any  $u \in P$ ,  $t \mapsto (\exp tX)(u)$  is a horizontal curve with respect to  $\omega$ , where  $\exp$  is the exponential map of  $H$ . This connection  $\omega$  is called the *canonical connection* of  $P$  associated with the reductive decomposition  $\mathfrak{h} = \mathfrak{l} + \mathfrak{m}$ .

### 3 Proof of Theorems A~F

In this section, we shall first prove Theorems A~F. We use the notations in Introduction. Let  $H \curvearrowright G/K$  be a Hermann action and  $Z_0$  be an element of  $\mathfrak{b}$ . Set  $M := H(\text{Exp } Z_0)$ .

*Proof of Theorem A.* Denote by  $\mathcal{H}$  the mean curvature vector of  $M$ . From (2.1) and (2.3), we have

$$\begin{aligned} \langle \mathcal{H}_{\text{Exp } Z_0}, \rho_{Z_0}^S(h)(g_{0*}v) \rangle &= - \sum_{i=1}^r \sum_{\beta \in \Delta'_+{}^V \setminus \Delta'_+{}^V_{Z_0}} \frac{n_i^\beta m_\beta^V}{\tan \beta(Z_0)} \beta_i(v) \\ &\quad + \sum_{i=1}^r \sum_{\beta \in \Delta'_+{}^H \setminus \Delta'_+{}^H_{Z_0}} n_i^\beta m_\beta^H \tan \beta(Z_0) \beta_i(v) \end{aligned}$$

for any  $v \in \mathfrak{b}$  and any  $h \in L$ . Hence,  $\mathcal{H}_{\text{Exp } Z_0}$  vanishes if and only if the following relations hold:

$$(3.1) \quad \sum_{\beta \in \Delta'_+{}^V \setminus \Delta'_+{}^V_{Z_0}} \frac{n_i^\beta m_\beta^V}{\tan \beta(Z_0)} = \sum_{\beta \in \Delta'_+{}^H \setminus \Delta'_+{}^H_{Z_0}} n_i^\beta m_\beta^H \tan \beta(Z_0) \quad (i = 1, \dots, r).$$

Since  $Z_0$  satisfies the condition (I) or (II) in Theorem A, (3.1) holds, that is,  $\mathcal{H}_{\text{Exp } Z_0}$  vanishes. Therefore  $M$  is minimal.

Next we shall show that there exists a reductive decomposition  $\mathfrak{h} = \mathfrak{l} + \mathfrak{m}$  with  $B_{\mathfrak{g}}(\mathfrak{l}, \mathfrak{m}) = 0$ . Easily we have

$$(3.2) \quad \mathfrak{l} = \mathfrak{z}_{\mathfrak{k} \cap \mathfrak{h}}(\mathfrak{b}) + \sum_{\beta \in \Delta'_+{}^V_{Z_0}} (\mathfrak{k}_\beta \cap \mathfrak{h}) + \sum_{\beta \in \Delta'_+{}^H_{Z_0}} (\mathfrak{p}_\beta \cap \mathfrak{h}).$$

Define a subspace  $\mathfrak{m}$  of  $\mathfrak{h}$  by

$$(3.3) \quad \mathfrak{m} := \mathfrak{z}_{\mathfrak{p} \cap \mathfrak{h}}(\mathfrak{b}) + \sum_{\beta \in \Delta'_+{}^V \setminus \Delta'_+{}^V_{Z_0}} (\mathfrak{k}_\beta \cap \mathfrak{h}) + \sum_{\beta \in \Delta'_+{}^H \setminus \Delta'_+{}^H_{Z_0}} (\mathfrak{p}_\beta \cap \mathfrak{h}).$$

Easily we can show that  $\mathfrak{h} = \mathfrak{l} + \mathfrak{m}$  is a reductive decomposition and that  $B_{\mathfrak{g}}(\mathfrak{l}, \mathfrak{m}) = 0$ .

Next we shall show that  $\nabla^{\omega_{\mathfrak{m}}} = \nabla^\perp$ . Take  $v \in \mathfrak{b} (\subset g_{0*}^{-1} T_{\text{Exp } Z_0}^\perp M)$ . Set  $g_s := \exp(1-s)Z_0$ . Let  $Z : [0, 1] \rightarrow \mathfrak{b}$  be a  $C^\infty$ -curve such that  $Z(0) = Z_0$  and that  $Z((0, 1])$  is contained in a fundamental domain of the Coxeter group associated with the principal  $H$ -orbit at an intersection point of the orbit and  $\mathfrak{b}$ . Set  $M_s := H(\text{Exp } Z(1-s))$  ( $0 \leq s \leq 1$ ). Denote by  $A^s$  the shape tensor of  $M_s$  and  $\widetilde{\nabla}$  the

Levi-Civita connection of  $G/K$ . Let  $\tilde{v}^s$  be the  $H$ -equivariant normal vector field of  $M_s$  ( $0 \leq s < 1$ ) arising from  $g_{s*}v$ . Since  $M_s$  ( $0 \leq s < 1$ ) is a principal orbit of a Hermann (hence hyperpolar) action,  $\tilde{v}^s$  is well-defined and it is a parallel normal vector field with respect to  $\nabla^\perp$ . Take  $X \in \mathfrak{k}_\beta \cap \mathfrak{h} (\subset \mathfrak{m})$  ( $\beta \in \Delta'_+{}^V \setminus \Delta'_{Z_0}{}^V$ ). Then, by using (2.3), we have

$$\tilde{\nabla}_{X_{\text{Exp } Z(1-s)}^*} \tilde{v}^s = -A_v^s X_{\text{Exp } Z(1-s)}^* = \frac{\beta(v)}{\tan \beta(Z_0)} X_{\text{Exp } Z(1-s)}^*.$$

and hence

$$\tilde{\nabla}_{X_{\text{Exp } Z_0}^*} (\exp tX)_{*\text{Exp}(Z_0)}(v) = \lim_{s \rightarrow 1-0} \tilde{\nabla}_{X_{\text{Exp } Z(1-s)}^*} \tilde{v}^s = \frac{\beta(v)}{\tan \beta(Z_0)} X_{\text{Exp } Z_0}^* \in T_{\text{Exp } Z_0} M.$$

Hence we obtain  $\nabla_{\hat{X}_{\text{Exp } Z_0}^*}^\perp (\exp tX)_{*\text{Exp}(Z_0)}(v) = 0$ . Take  $Y \in \mathfrak{p}_\beta \cap \mathfrak{h} (\subset \mathfrak{m})$  ( $\beta \in \Delta'_+{}^H \setminus \Delta'_{Z_0}{}^H$ ). Then, by using (2.3), we have

$$\tilde{\nabla}_{Y_{\text{Exp } Z(1-s)}^*} \tilde{v}^s = -A_v^s Y_{\text{Exp } Z(1-s)}^* = -\beta(v) \tan \beta(Z_0) Y_{\text{Exp } Z(1-s)}^*.$$

and hence

$$\begin{aligned} \tilde{\nabla}_{Y_{\text{Exp } Z_0}^*} (\exp tY)_{*\text{Exp } Z_0}(v) &= \lim_{s \rightarrow 1-0} \tilde{\nabla}_{Y_{\text{Exp } Z(1-s)}^*} \tilde{v}^s \\ &= -\beta(v) \tan \beta(Z_0) Y_{\text{Exp } Z_0}^* \in T_{\text{Exp } Z_0} M. \end{aligned}$$

Hence we obtain  $\nabla_{\hat{Y}_{\text{Exp } Z_0}^*}^\perp (\exp tY)_{*\text{Exp}(Z_0)}(v) = 0$ . Therefore, it follows from the arbitrariness of  $X, Y$  and  $\beta$  that  $t \mapsto (\exp t\hat{X})_{*\text{Exp } Z_0}(v)$  is  $\nabla^\perp$ -parallel along  $t \mapsto (\exp t\hat{X})(\text{Exp } Z_0)$  for any  $\hat{X} \in \mathfrak{m}$ . Take any  $h \in L$ . Similarly we can show that  $t \mapsto (\exp t\hat{X})_{*\text{Exp } Z_0}(\rho_{Z_0}^S(h)(g_{0*}v))$  is  $\nabla^\perp$ -parallel along  $t \mapsto (\exp t\hat{X})(\text{Exp } Z_0)$  for any  $\hat{X} \in \mathfrak{m}$ . Note that this fact has been showed in [IST] in different method. On the other hand, it follows from the definition of  $\omega$  that  $t \mapsto (\exp t\hat{X})_{*\text{Exp } Z_0}(\rho_{Z_0}^S(h)(g_{0*}v))$  is  $\nabla^{\omega_{\mathfrak{m}}}$ -parallel along  $t \mapsto (\exp t\hat{X})(\text{Exp } Z_0)$  for any  $\hat{X} \in \mathfrak{m}$ . Therefore we obtain  $\nabla^\perp = \nabla^{\omega_{\mathfrak{m}}}$ . The statement for  $\bigcap_{v \in T_x^\perp M} \text{Ker } A_v$  follows from (2.3) directly.

q.e.d.

Next we prove Theorem B.

*Proof of Theorem B.* This statement of this theorem follows from (2.3) directly.

q.e.d.

Next we prove Theorems C~F.

*Proof of Theorems C~F.* Define a diffeomorphism  $\psi : H/L \rightarrow M$  by  $\psi(hL) := h \cdot \text{Exp } Z_0$  ( $h \in H$ ). Next we shall show that  $(\psi^* g_I)_{eL} = cB_{\mathfrak{g}}|_{\mathfrak{m} \times \mathfrak{m}}$ , where

$$c = \begin{cases} \frac{3}{4} & (\text{in case of Theorems C}) \\ \frac{1}{4} & (\text{in case of Theorem D}) \\ \frac{1}{2} & (\text{in case of Theorem E}) \\ 1 & (\text{in case of Theorem F}). \end{cases}$$

In the sequel, we omit the notation  $\psi^*$ . For each  $X \in \mathfrak{m} (= T_{eL}(H/L) = T_{\text{Exp } Z_0} M)$ , denote by  $X^*$  the Killing field on  $M$  associated with  $X$ , that is,  $X_p^* := \frac{d}{dt}|_{t=0}(\exp tX)(p)$  ( $p \in M$ ). From the definition of  $\psi$ , we have  $\psi_{*eL} X = X_{\text{Exp } Z_0}^*$ . Take  $S_{\beta_1} \in \mathfrak{f}_{\beta_1} \cap \mathfrak{h}$  ( $\beta_1 \in \Delta'_+{}^H \setminus \Delta'_{Z_0}{}^H$ ) and  $\hat{S}_{\beta_2} \in \mathfrak{p}_{\beta_2} \cap \mathfrak{h}$  ( $\beta_2 \in \Delta'_+{}^V \setminus \Delta'_{Z_0}{}^V$ ). Let  $T_{\beta_1}$  be the element of  $\mathfrak{p}_{\beta_1} \cap \mathfrak{q}$  such that  $\text{ad}(b)(S_{\beta_1}) = \beta_1(b)T_{\beta_1}$  for any  $b \in \mathfrak{b}$ . Then we have

$$(3.4) \quad \psi_{*eL}(S_{\beta_1}) = (S_{\beta_1}^*)_{\text{Exp } Z_0} = -\sin \beta_1(Z_0)(\exp Z_0)_*(T_{\beta_1})$$

and

$$(3.5) \quad \psi_{*eL}(\hat{S}_{\beta_2}) = (\hat{S}_{\beta_2}^*)_{\text{Exp } Z_0} = \cos \beta_2(Z_0)(\exp Z_0)_*(\hat{S}_{\beta_2}).$$

Hence, since  $H$  and  $Z_0$  is as in Theorems C~F, we have  $(g_I)_{eL}(S_{\beta_1}, S_{\beta_1}) = cB_{\mathfrak{g}}(S_{\beta_1}, S_{\beta_1})$  and  $(g_I)_{eL}(\hat{S}_{\beta_2}, \hat{S}_{\beta_2}) = cB_{\mathfrak{g}}(\hat{S}_{\beta_2}, \hat{S}_{\beta_2})$ . If the cohomogeneity of the  $H$ -action is equal to the rank of  $G/K$ , then we have  $\mathfrak{z}_{\mathfrak{p} \cap \mathfrak{h}}(\mathfrak{b}) = 0$ . Therefore we obtain  $(g_I)_{eL} = cB_{\mathfrak{g}}|_{\mathfrak{m} \times \mathfrak{m}}$ . Also, in Theorems C~E,  $\bigcap_{v \in T_x^\perp M} \text{Ker } A_v = \{0\}$  follows from the statement for  $\bigcap_{v \in T_x^\perp M} \text{Ker } A_v$  in Theorem A directly. q.e.d.

## 4 Examples

In this section, we give examples of a Hermann action  $H \curvearrowright G/K$  and  $Z_0 \in \tilde{C}$  as in Theorems B, C and F. We use the notations in Introduction.

*Example 1.* We consider the isotropy action of  $SU(3n+3)/SO(3n+3)$ . Then we have  $\Delta_+ = \Delta'_+ = \Delta'^V_+$  (which is of  $(\mathfrak{a}_{3n+2})$ -type) and  $\Delta'^H_+ = \emptyset$ . Let  $\Pi = \{\beta_1, \dots, \beta_{3n+2}\}$  be a simple root system of  $\Delta'_+$ , where we order  $\beta_1, \dots, \beta_{3n+2}$  as the Dynkin diagram of  $\Delta'_+$  is as in Fig. 1,  $\Delta'_+ = \{\beta_i + \dots + \beta_j \mid 1 \leq i, j \leq 3n+2\}$ . For any  $\beta \in \Delta'_+$ , we have  $m_\beta = 1$ . Let  $Z_0$  be the point of  $\mathfrak{b}$  defined by  $\beta_{n+1}(Z_0) = \beta_{2n+2}(Z_0) = \frac{\pi}{3}$  and  $\beta_i(Z_0) = 0$  ( $i \in \{1, \dots, 3n+2\} \setminus \{n+1, 2n+2\}$ ). Clearly we have  $m_\beta^V = 1$ ,



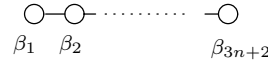
$m_\beta^H = 0$  and  $\beta(Z_0) \equiv 0, \frac{\pi}{3}$  or  $\frac{2\pi}{3} \pmod{\pi}$  for any  $\beta \in \Delta'_+$ . For simplicity, set  $\beta_{ij} := \beta_i + \cdots + \beta_j$  ( $1 \leq i \leq j \leq 3n+2$ ). Easily we can show

$$\begin{aligned} & \{\beta \in \Delta'_+ \mid \beta(Z_0) \equiv \frac{\pi}{3} \pmod{\pi}\} \\ &= \{\beta_{ij} \mid 1 \leq i \leq n+1 \leq j < 2n+2, \text{ or } n+1 < i \leq 2n+2 \leq j \leq 3n+2\} \end{aligned}$$

and

$$\begin{aligned} & \{\beta \in \Delta'_+ \mid \beta(Z_0) \equiv \frac{2\pi}{3} \pmod{\pi}\} \\ &= \{\beta_{ij} \mid 1 \leq i \leq n+1, 2n+2 \leq j \leq 3n+2\}. \end{aligned}$$

From these facts, it follows that the condition (I) holds. Thus  $Z_0$  is as in the statement of Theorem C. Also, it is easy to show that  $M$  is not austere.



**Figure 1.**

*Example 2.* We consider the isotropy action of  $SU(6n+6)/Sp(3n+3)$ . Then we have  $\Delta_+ = \Delta'_+ = \Delta'^V_+$  (which is of  $(\mathfrak{a}_{3n+2})$ -type) and  $\Delta'^H_+ = \emptyset$ . Let  $\Pi = \{\beta_1, \dots, \beta_{3n+2}\}$  be a simple root system of  $\Delta'_+$ , where we order  $\beta_1, \dots, \beta_{3n+2}$  as above. We have  $m_\beta = 4$  for any  $\beta \in \Delta'_+$ . Let  $Z_0$  be the point of the closure of  $\mathfrak{b}$  defined by  $\beta_{n+1}(Z_0) = \beta_{2n+2}(Z_0) = \frac{\pi}{3}$  and  $\beta_i(Z_0) = 0$  ( $i \in \{1, \dots, 3n+2\} \setminus \{n+1, 2n+2\}$ ). Clearly we have  $m_\beta^V = 4$ ,  $m_\beta^H = 0$  and  $\beta(Z_0) \equiv 0, \frac{\pi}{3}$  or  $\frac{2\pi}{3} \pmod{\pi}$  for any  $\beta \in \Delta'_+$ . For simplicity, set  $\beta_{ij} := \beta_i + \cdots + \beta_j$  ( $1 \leq i \leq j \leq 3n+2$ ). Easily we can show

$$\begin{aligned} & \{\beta \in \Delta'_+ \mid \beta(Z_0) \equiv \frac{\pi}{3} \pmod{\pi}\} \\ &= \{\beta_{ij} \mid 1 \leq i \leq n+1 \leq j < 2n+2, \text{ or } n+1 < i \leq 2n+2 \leq j \leq 3n+2\} \end{aligned}$$

and

$$\begin{aligned} & \{\beta \in \Delta'_+ \mid \beta(Z_0) \equiv \frac{2\pi}{3} \pmod{\pi}\} \\ &= \{\beta_{ij} \mid 1 \leq i \leq n+1, 2n+2 \leq j \leq 3n+2\}. \end{aligned}$$

From these facts, it follows that the condition (I) holds. Thus  $Z_0$  is as in the statement of Theorem C. Also, it is easy to show that  $M$  is not austere.

*Example 3.* We consider the isotropy action of  $SU(3)/S(U(1) \times U(2))$  (2-dimensional complex projective space). Then we have  $\Delta_+ = \Delta'_+ = \Delta'^V_+ = \{\beta, 2\beta\}$  and  $\Delta'^H_+ = \emptyset$ ,  $m_\beta = 2$  and  $m_{2\beta} = 1$ . Let  $Z_0$  be the point of  $\mathfrak{b}$  defined by  $\beta(Z_0) = \frac{\pi}{3}$ . Clearly  $Z_0$  satisfies the condition (I). Thus  $Z_0$  is as in the statement of Theorem C. Also, it is easy to show that  $M$  is not austere.

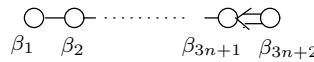
*Example 4.* We consider the isotropy action of  $Sp(3n+2)/U(3n+2)$ . Then we have  $\Delta_+ = \Delta'_+ = \Delta'^V_+$  (which is of  $(\mathfrak{c}_{3n+2})$ -type) and  $\Delta'^H_+ = \emptyset$ . Let  $\Pi = \{\beta_1, \dots, \beta_{3n+2}\}$  be a simple root system of  $\Delta'_+$ , where we order  $\beta_1, \dots, \beta_{3n+2}$  as the Dynkin diagram of  $\Delta'_+$  is as in Fig. 2. We have  $m_\beta = 1$  for any  $\beta \in \Delta'_+$ . Let  $Z_0$  be the point of  $\mathfrak{b}$  defined by  $\beta_{n+1}(Z_0) = \beta_{2n+2}(Z_0) = \beta_{3n+2}(Z_0) = \frac{\pi}{3}$  and  $\beta_i(Z_0) = 0$  ( $i \in \{1, \dots, 3n+2\} \setminus \{n+1, 2n+2, 3n+2\}$ ). Clearly we have  $m^V_\beta = 1$ ,  $m^H_\beta = 0$  and  $\beta(Z_0) \equiv 0, \frac{\pi}{3}$  or  $\frac{2\pi}{3} \pmod{\pi}$  for any  $\beta \in \Delta'_+$ . For simplicity, set  $\beta_{ij} := \beta_i + \dots + \beta_j$  ( $1 \leq i \leq j \leq 3n+2$ ),  $\widehat{\beta}_i := 2(\beta_i + \dots + \beta_{3n+1}) + \beta_{3n+2}$  and  $\widehat{\beta}_{ij} := \beta_i + \dots + \beta_{j-1} + 2(\beta_j + \dots + \beta_{3n+1}) + \beta_{3n+2}$  ( $1 \leq i < j \leq 3n+1$ ). Easily we can show

$$\begin{aligned} & \{\beta \in \Delta'^V_+ \mid \beta(Z_0) \equiv \frac{\pi}{3} \pmod{\pi}\} \\ &= \{\beta_{ij} \mid 1 \leq i \leq n+1 \leq j < 2n+2 \text{ or } n+1 < i \leq 2n+2 \leq j < 3n+2 \\ & \quad \text{or } 2n+3 \leq i \leq j = 3n+2\} \\ & \cup \{\widehat{\beta}_i \mid 2n+3 \leq i \leq 3n+1\} \\ & \cup \{\widehat{\beta}_{ij} \mid 2n+3 \leq i < j \leq 3n+1 \text{ or } 1 \leq i \leq n+1 < j \leq 2n+2\} \end{aligned}$$

and

$$\begin{aligned} & \{\beta \in \Delta'^V_+ \mid \beta(Z_0) \equiv \frac{2\pi}{3} \pmod{\pi}\} \\ &= \{\beta_{ij} \mid "1 \leq i \leq n+1 \ \& \ 2n+2 \leq j \leq 3n+1" \text{ or} \\ & \quad "n+2 \leq i \leq 2n+2 \ \& \ j = 3n+2"\} \\ & \cup \{\widehat{\beta}_i \mid 1 \leq i \leq n+1\} \\ & \cup \{\widehat{\beta}_{ij} \mid 1 \leq i < j \leq n+1 \text{ or } n+2 \leq i \leq 2n+2 < j \leq 3n+1\}. \end{aligned}$$

From these facts, it follows that the condition (I) holds. Thus  $Z_0$  is as in the statement of Theorem C. Also, it is easy to show that  $M$  is not austere.



**Figure 2.**

By referring Tables 1 and 2 in [K2], we shall list up Hermann actions of cohomogeneity two on irreducible symmetric spaces of compact type and rank two satisfying

$$(i) \ m^V_\beta = m^H_\beta \ (\forall \beta \in \Delta'_+) \quad \text{or} \quad (ii) \ \Delta'^V_+ \cap \Delta'^H_+ = \emptyset.$$

All of such Hermann actions satisfying (i) are as in Table 1. In Table 1,  $\beta_{(m)}$  means

$m^V_\beta = m^H_\beta = m$ . All of such Hermann actions satisfying (ii) are the dual actions (see Table 3) of Hermann actions on symmetric spaces of non-compact type as in

Table 2. In Table 3,  $\beta_{(m)}$  means  $m_{\beta}^V$  or  $m_{\beta}^H$  is equal to  $m$ . Since the Hermann actions in Table 2 are commutative, so are also the Hermann actions in Table 3. Also, since  $\Delta'_+{}^V \cap \Delta'_+{}^H = \emptyset$  as in Table 3 and  $G/K$  is irreducible, there exists an inner automorphism  $\rho$  of  $G$  with  $\rho(K) = H$  by Proposition 4.39 in [I]. According to the proof of the proposition,  $\rho$  is given explicitly by  $\rho = \text{Ad}_G(\exp b)$ , where  $\text{Ad}_G$  is the adjoint representation of  $G$  and  $b$  is the element of  $\mathfrak{b}$  satisfying

$$(\beta_1(b), \beta_2(b)) = \begin{cases} (0, \frac{\pi}{2}) & (\text{in case of } (1), (2), (3), (4), (6), (9), (10), (11)) \\ (\frac{\pi}{2}, 0) & (\text{in case of } (5), (7)) \\ (\frac{\pi}{2}, \frac{\pi}{2}) & (\text{in case of } (8)). \end{cases}$$

$H \curvearrowright G/K$	$\Delta'_+{}^V = \Delta'_+{}^H$
$SO(6) \curvearrowright SU(6)/Sp(3)$	$\{\beta_1, \beta_2, \beta_1 + \beta_2\}$ (2) (2) (2)
$SO(2)^2 \times SO(3)^2 \curvearrowright (SO(5) \times SO(5))/SO(5)$	$\{\beta_1, \beta_2, \beta_1 + \beta_2, 2\beta_1 + \beta_2\}$ (1) (1) (1) (1)
$SU(2)^2 \cdot SO(2)^2 \curvearrowright (Sp(2) \times Sp(2))/Sp(2)$	$\{\beta_1, \beta_2, \beta_1 + \beta_2, 2\beta_1 + \beta_2\}$ (1) (1) (1) (1)
$Sp(4) \curvearrowright E_6/F_4$	$\{\beta_1, \beta_2, \beta_1 + \beta_2\}$ (4) (4) (4)
$SU(2)^4 \curvearrowright (G_2 \times G_2)/G_2$	$\{\beta_1, \beta_2, \beta_1 + \beta_2, 2\beta_1 + \beta_2, 3\beta_1 + \beta_2, 3\beta_1 + 2\beta_2\}$ (1) (1) (1) (1) (1) (1)

**Table 1.**

(1)	$SO_0(1, 2) \curvearrowright SL(3, \mathbb{R})/SO(3)$
(2)	$Sp(1, 2) \curvearrowright SU^*(6)/Sp(3)$
(3)	$U(2, 3) \curvearrowright SO^*(10)/U(5)$
(4)	$SO_0(2, 3) \curvearrowright SO(5, \mathbb{C})/SO(5)$
(5)	$U(1, 1) \curvearrowright Sp(2, \mathbb{R})/U(2)$
(6)	$Sp(2, \mathbb{R}) \curvearrowright Sp(2, \mathbb{C})/Sp(2)$
(7)	$Sp(1, 1) \curvearrowright Sp(2, \mathbb{C})/Sp(2)$
(8)	$SO^*(10) \cdot U(1) \curvearrowright E_6^{-14}/Spin(10) \cdot U(1)$
(9)	$F_4^{-20} \curvearrowright E_6^{-26}/F_4$
(10)	$SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \curvearrowright G_2^2/SO(4)$
(11)	$G_2^2 \curvearrowright G_2^C/G_2$

**Table 2.**

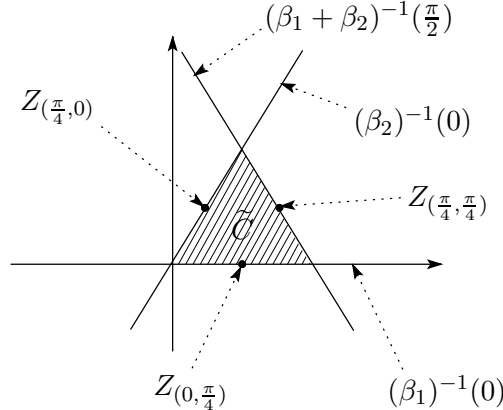
	$H \curvearrowright G/K$	$\Delta'_+{}^V$	$\Delta'_+{}^H$
(1)	$SO_0(1, 2)^* \curvearrowright SU(3)/SO(3)$	$\{\beta_1\}_{(1)}$	$\{\beta_2, \beta_1 + \beta_2\}_{(1) \quad (1)}$
(2)	$Sp(1, 2)^* \curvearrowright SU(6)/Sp(3)$	$\{\beta_1\}_{(4)}$	$\{\beta_2, \beta_1 + \beta_2\}_{(4) \quad (4)}$
(3)	$U(2, 3)^* \curvearrowright SO(10)/U(5)$	$\{\beta_1, 2\beta_1, 2\beta_1 + 2\beta_2\}_{(4) \quad (1) \quad (1)}$	$\{\beta_2, \beta_1 + \beta_2, 2\beta_1 + \beta_2\}_{(4) \quad (4) \quad (4)}$
(4)	$SO_0(2, 3)^* \curvearrowright (SO(5) \times SO(5))/SO(5)$	$\{\beta_1\}_{(2)}$	$\{\beta_2, \beta_1 + \beta_2, 2\beta_1 + \beta_2\}_{(2) \quad (2) \quad (2)}$
(5)	$U(1, 1)^* \curvearrowright Sp(2)/U(2)$	$\{\beta_2, 2\beta_1 + \beta_2\}_{(1) \quad (1)}$	$\{\beta_1, \beta_1 + \beta_2\}_{(1) \quad (1)}$
(6)	$Sp(2, \mathbb{R})^* \curvearrowright (Sp(2) \times Sp(2))/Sp(2)$	$\{\beta_1\}_{(2)}$	$\{\beta_2, \beta_1 + \beta_2, 2\beta_1 + \beta_2\}_{(2) \quad (2) \quad (2)}$
(7)	$Sp(1, 1)^* \curvearrowright (Sp(2) \times Sp(2))/Sp(2)$	$\{\beta_2, 2\beta_1 + \beta_2\}_{(2) \quad (2)}$	$\{\beta_1, \beta_1 + \beta_2\}_{(2) \quad (2)}$
(8)	$(SO^*(10) \cdot U(1))^* \curvearrowright E_6/Spin(10) \cdot U(1)$	$\{\beta_1, 2\beta_1, 2\beta_1 + 2\beta_2\}_{(8) \quad (1) \quad (1)}$	$\{\beta_2, \beta_1 + \beta_2, 2\beta_1 + \beta_2\}_{(6) \quad (9) \quad (5)}$
(9)	$(F_4^{-20})^* \curvearrowright E_6/F_4$	$\{\beta_1\}_{(8)}$	$\{\beta_2, \beta_1 + \beta_2\}_{(8) \quad (8)}$
(10)	$(SL(2, \mathbb{R}) \times SL(2, \mathbb{R}))^* \curvearrowright G_2/SO(4)$	$\{\beta_1, 3\beta_1 + 2\beta_2\}_{(1) \quad (1)}$	$\{\beta_2, \beta_1 + \beta_2, 2\beta_1 + \beta_2, 3\beta_1 + \beta_2\}_{(1) \quad (1) \quad (1) \quad (1)}$
(11)	$(G_2^2)^* \curvearrowright (G_2 \times G_2)/G_2$	$\{\beta_1, 3\beta_1 + 2\beta_2\}_{(2) \quad (2)}$	$\{\beta_2, \beta_1 + \beta_2, 2\beta_1 + \beta_2, 3\beta_1 + \beta_2\}_{(2) \quad (2) \quad (2) \quad (2)}$

**Table 3.**

According to Theorem B, we obtain the following fact.

**Proposition 4.1.** *Let  $H \curvearrowright G/K$  be a Hermann action in Table 1 and  $Z_0$  an element of  $\mathfrak{h}$  satisfying  $(\beta_1(Z_0), \beta_2(Z_0)) = (0, \frac{\pi}{4}), (\frac{\pi}{4}, 0)$  or  $(\frac{\pi}{4}, \frac{\pi}{4})$ . Then  $M = H(\text{Exp } Z_0)$  is a (non-totally geodesic) austere submanifold.*

Denote by  $Z_{(a,b)}$  the element  $Z$  of  $\mathfrak{h}$  satisfying  $(\beta_1(Z), \beta_2(Z)) = (a, b)$ . In the case where  $\Delta'$  is of type  $(a_2)$ , three points of  $\mathfrak{h}$  as in Proposition 4.1 are as in Figure 3.



**Figure 3.**

**Proposition 4.2.** *Let  $H \curvearrowright G/K$  be a Hermann action in Table 3 and  $Z_0$  an element of the closure of  $\tilde{C}(\subset \mathfrak{b})$  such that  $H(\text{Exp } Z_0)$  is minimal. Then, as in Tables 4 ~ 13,  $Z_0$  satisfies the condition in Theorem C or F, or it does not satisfy the conditions in Theorems C~F.*

*Remark 4.1.* There exist exactly seven elements  $Z_0$  of the closure of  $\tilde{C}(\subset \mathfrak{b})$  such that  $H(\text{Exp } Z_0)$  is minimal.

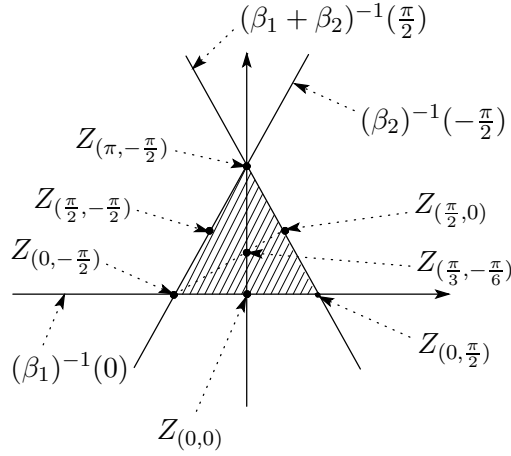
$(a, b)$	$Z_{(a,b)}$	$M = SO_0(1, 2)^*(\text{Exp } Z_{(a,b)})$	$\dim M$
$(0, -\frac{\pi}{2})$	as in Theorem F	one-point set	0
$(0, \frac{\pi}{2})$	as in Theorem F	one-point set	0
$(\pi, -\frac{\pi}{2})$	as in Theorem F	one-point set	0
$(0, 0)$	as in Theorem F	totally geodesic	2
$(\frac{\pi}{2}, 0)$	as in Theorem F	totally geodesic	2
$(\frac{\pi}{2}, -\frac{\pi}{2})$	as in Theorem F	totally geodesic	2
$(\frac{\pi}{3}, -\frac{\pi}{6})$	as in Theorem C	not austere	3

$$SO_0(1, 2)^* \curvearrowright SU(3)/SO(3)$$

$$(\dim SU(3)/SO(3) = 5)$$

**Table 4.**

The positions of  $Z_0$ 's in Table 4 are as in Figure 4.



**Figure 4.**

$(a, b)$	$Z_{(a,b)}$	$M = Sp(1, 2)^*(\text{Exp } Z_{(a,b)})$	$\dim M$
$(0, -\frac{\pi}{2})$	as in Theorem F	one-point set	0
$(0, \frac{\pi}{2})$	as in Theorem F	one-point set	0
$(\pi, -\frac{\pi}{2})$	as in Theorem F	one-point set	0
$(0, 0)$	as in Theorem F	totally geodesic	8
$(\frac{\pi}{2}, 0)$	as in Theorem F	totally geodesic	8
$(\frac{\pi}{2}, -\frac{\pi}{2})$	as in Theorem F	totally geodesic	8
$(\frac{\pi}{3}, -\frac{\pi}{6})$	as in Theorem C	not austere	12

$$Sp(1, 2)^* \curvearrowright SU(6)/Sp(3)$$

$$(\dim SU(6)/Sp(3) = 14)$$

**Table 5.**

The positions of  $Z_0$ 's in Table 5 are as in Figure 4.

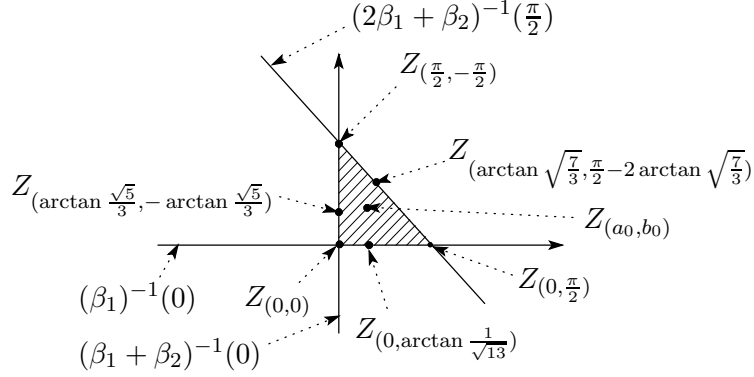
$(a, b)$	$Z_{(a,b)}$	$M = U(2, 3)^*(\text{Exp } Z_{(a,b)})$	$\dim M$
$(0, \frac{\pi}{2})$	as in Theorem F	one-point set	0
$(0, 0)$	as in Theorem F	totally geodesic	12
$(\frac{\pi}{2}, -\frac{\pi}{2})$	as in Theorem F	totally geodesic	8
$(\arctan \sqrt{\frac{7}{3}}, \frac{\pi}{2} - \arctan \sqrt{\frac{7}{3}})$	not as in Theorems C~F	not austere	14
$(0, \arctan \frac{1}{\sqrt{13}})$	not as in Theorems C~F	not austere	13
$(\arctan \frac{\sqrt{5}}{3}, -\arctan \frac{\sqrt{5}}{3})$	not as in Theorems C~F	not austere	17
$(a_0, b_0)$	not as in Theorems C~F	not austere	18

$$U(2, 3)^* \curvearrowright SO(10)/U(5)$$

$$(\dim SO(10)/U(5) = 20)$$

**Table 6.**

The positions of  $Z_0$ 's in Table 6 are as in Figure 5. Also, the numbers  $a_0$  and  $b_0$  in Table 6 are real numbers such that  $a_0, b_0 \not\equiv \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{3\pi}{4} \pmod{\pi}$ .



**Figure 5.**

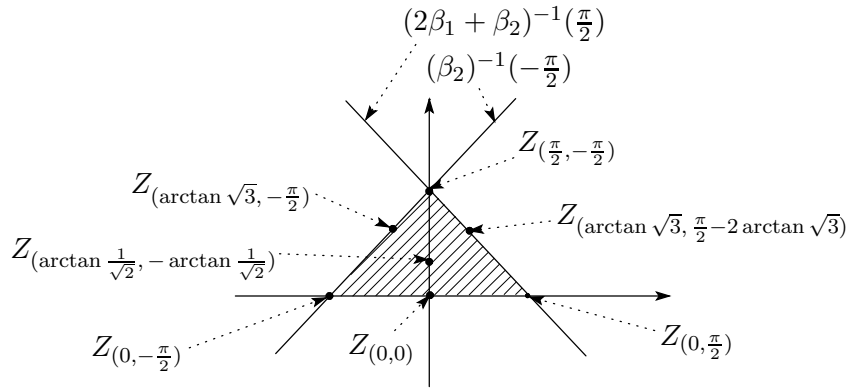
$(a, b)$	$Z_{(a,b)}$	$M = SO_0(2, 3)^*(\text{Exp } Z_{(a,b)})$	$\dim M$
$(0, -\frac{\pi}{2})$	as in Theorem F	one-point set	0
$(0, \frac{\pi}{2})$	as in Theorem F	one-point set	0
$(\frac{\pi}{2}, -\frac{\pi}{2})$	as in Theorem F	totally geodesic	4
$(0, 0)$	as in Theorem F	totally geodesic	6
$(\arctan \sqrt{3}, -\frac{\pi}{2})$	not as in Theorems C~F	not austere	6
$(\arctan \sqrt{3}, \frac{\pi}{2} - 2 \arctan \sqrt{3})$	not as in Theorems C~F	not austere	6
$(\arctan \frac{1}{\sqrt{2}}, -\arctan \frac{1}{\sqrt{2}})$	not as in Theorems C~F	not austere	8

$$SO_0(2, 3)^* \curvearrowright (SO(5) \times SO(5))/SO(5)$$

$$(\dim(SO(5) \times SO(5))/SO(5) = 10)$$

**Table 7.**

The positions of  $Z_0$ 's in Table 7 are as in Figure 6.



**Figure 6.**

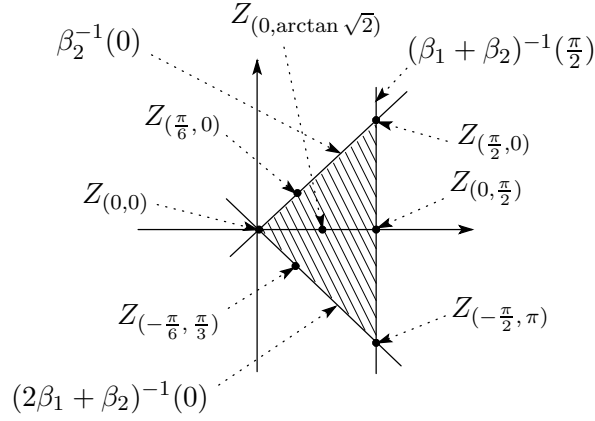
$(a, b)$	$Z_{(a,b)}$	$M = U(1, 1)^*(\text{Exp } Z_{(a,b)})$	$\dim M$
$(\frac{\pi}{2}, 0)$	as in Theorem F	one-point set	0
$(-\frac{\pi}{2}, \pi)$	as in Theorem F	one-point set	0
$(0, 0)$	as in Theorem F	totally geodesic	2
$(\frac{\pi}{6}, 0)$	as in Theorem C	not austere	3
$(-\frac{\pi}{6}, \frac{\pi}{3})$	as in Theorem C	not austere	3
$(0, \frac{\pi}{2})$	as in Theorem F	totally geodesic	3
$(0, \arctan \sqrt{2})$	not as in Theorems C~F	not austere	4

$$U(1, 1)^* \curvearrowright Sp(2)/U(2)$$

$$(\dim Sp(2)/U(2) = 6)$$

**Table 8.**

The positions of  $Z_0$ 's in Table 8 are as in Figure 7.



**Figure 7.**



$(a, b)$	$Z_{(a,b)}$	$M = Sp(2, \mathbb{R})^*(\text{Exp } Z_{(a,b)})$	$\dim M$
$(0, -\frac{\pi}{2})$	as in Theorem F	one-point set	0
$(0, \frac{\pi}{2})$	as in Theorem F	one-point set	0
$(\frac{\pi}{2}, -\frac{\pi}{2})$	as in Theorem F	totally geodesic	4
$(0, 0)$	as in Theorem F	totally geodesic	6
$(\arctan \sqrt{3}, -\frac{\pi}{2})$	not as in Theorems C~F	not austere	6
$(\arctan \sqrt{3}, \frac{\pi}{2} - 2 \arctan \sqrt{3})$	not as in Theorems C~F	not austere	6
$(\arctan \frac{1}{\sqrt{2}}, -\arctan \frac{1}{\sqrt{2}})$	not as in Theorems C~F	not austere	8

$$Sp(2, \mathbb{R})^* \curvearrowright (Sp(2) \times Sp(2))/Sp(2)$$

$$(\dim (Sp(2) \times Sp(2))/Sp(2) = 10)$$

**Table 9.**

The positions of  $Z_0$ 's in Table 9 are as in Figure 6.

$(a, b)$	$Z_{(a,b)}$	$M = Sp(1, 1)^*(\text{Exp } Z_{(a,b)})$	$\dim M$
$(\frac{\pi}{2}, 0)$	as in Theorem F	one-point set	0
$(-\frac{\pi}{2}, \pi)$	as in Theorem F	one-point set	0
$(0, 0)$	as in Theorem F	totally geodesic	4
$(\frac{\pi}{6}, 0)$	as in Theorem C	not austere	6
$(-\frac{\pi}{6}, \frac{\pi}{3})$	as in Theorem C	not austere	6
$(0, \frac{\pi}{2})$	as in Theorem F	totally geodesic	6
$(0, \arctan \sqrt{2})$	not as in Theorems C~F	not austere	8

$$Sp(1, 1)^* \curvearrowright (Sp(2) \times Sp(2))/Sp(2)$$

$$(\dim (Sp(2) \times Sp(2))/Sp(2) = 10)$$

**Table 10.**

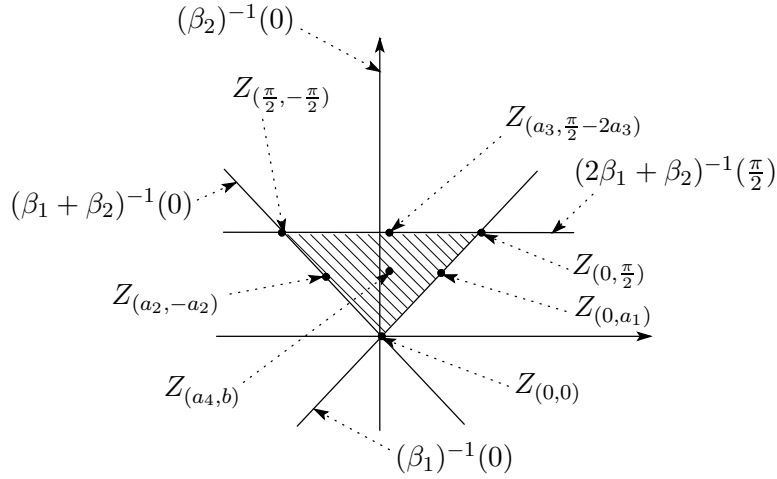
The positions of  $Z_0$ 's in Table 10 are as in Figure 7.

$(a, b)$	$Z_{(a,b)}$	$M = (SO^*(10) \cdot U(1))^*(\text{Exp } Z_{(a,b)})$	$\dim M$
$(0, 0)$	as in Theorem F	totally geodesic	20
$(0, \frac{\pi}{2})$	as in Theorem F	one-point set	0
$(\frac{\pi}{2}, -\frac{\pi}{2})$	as in Theorem F	totally geodesic	17
$(0, a_1)$	not as in Theorems C~F	not austere	21
$(a_2, -a_2)$	not as in Theorems C~F	not austere	29
$(a_3, \frac{\pi}{2} - 2a_3)$	not as in Theorems C~F	not austere	25
$(a_4, b)$	not as in Theorems C~F	not austere	30

$$\begin{aligned}
(SO^*(10) \cdot U(1))^* &\curvearrowright E_6/Spin(10) \cdot U(1) \\
(\dim E_6/Spin(10) \cdot U(1) &= 32)
\end{aligned}$$

**Table 11.**

The positions of  $Z_0$ 's in Table 11 are as in Figure 8. The numbers  $a_i$  ( $i = 1, 2, 3, 4$ ) and  $b$  in Table 11 are real numbers such that  $a_i, b \not\equiv \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{3\pi}{4} \pmod{\pi}$ .



**Figure 8.**

$(a, b)$	$Z_{(a,b)}$	$M = (F_4^{-20})^*(\text{Exp } Z_{(a,b)})$	$\dim M$
$(0, -\frac{\pi}{2})$	as in Theorem F	one-point set	0
$(0, \frac{\pi}{2})$	as in Theorem F	one-point set	0
$(\pi, -\frac{\pi}{2})$	as in Theorem F	one-point set	0
$(0, 0)$	as in Theorem F	totally geodesic	16
$(\frac{\pi}{2}, 0)$	as in Theorem F	totally geodesic	16
$(\frac{\pi}{2}, -\frac{\pi}{2})$	as in Theorem F	totally geodesic	16
$(\frac{\pi}{3}, -\frac{\pi}{6})$	as in Theorem C	not austere	24

$$(F_4^{-20})^* \curvearrowright E_6/F_4$$

$$(\dim E_6/F_4 = 26)$$

**Table 12.**

The positions of  $Z_0$ 's in Table 12 are as in Figure 4.

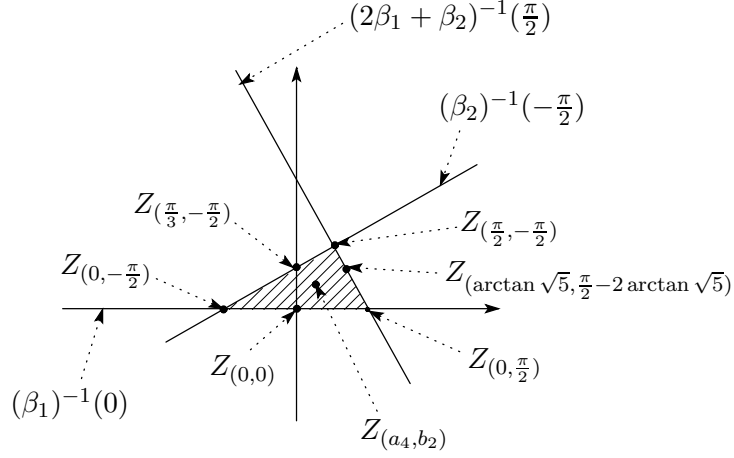
$(a, b)$	$Z_{(a,b)}$	$M = (SL(2, \mathbb{R}) \times SL(2, \mathbb{R}))^*(\text{Exp } Z_{(a,b)})$	$\dim M$
$(0, -\frac{\pi}{2})$	as in Theorem F	one-point set	0
$(0, \frac{\pi}{2})$	as in Theorem F	one-point set	0
$(\frac{\pi}{2}, -\frac{\pi}{2})$	as in Theorem F	totally geodesic	4
$(\frac{\pi}{3}, -\frac{\pi}{2})$	as in Theorem C	not austere	3
$(\arctan \sqrt{5}, \frac{\pi}{2} - 2 \arctan \sqrt{5})$	not as in Theorems C~F	not austere	5
$(a_4, b_2)$	not as in Theorems C~F	not austere	6

$$(SL(2, \mathbb{R}) \times SL(2, \mathbb{R}))^* \curvearrowright G_2/SO(4)$$

$$(\dim G_2/SO(4) = 8)$$

**Table 13.**

The positions of  $Z_0$ 's in Table 13 are as in Figure 9. The numbers  $a_4$  and  $b_2$  in Table 13 are real numbers such that  $a_4, b_2 \not\equiv \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{3\pi}{4} \pmod{\pi}$ .



**Figure 9.**

$(a, b)$	$Z_{(a,b)}$	$M = (G_2^2)^*(\text{Exp } Z_{(a,b)})$	$\dim M$
$(0, -\frac{\pi}{2})$	as in Theorem F	one-point set	0
$(0, \frac{\pi}{2})$	as in Theorem F	one-point set	0
$(\frac{\pi}{2}, -\frac{\pi}{2})$	as in Theorem F	totally geodesic	8
$(\frac{\pi}{3}, -\frac{\pi}{2})$	as in Theorem C	not austere	6
$(\arctan \sqrt{5}, \frac{\pi}{2} - 2 \arctan \sqrt{5})$	not as in Theorems C~F	not austere	10
$(a_5, b_3)$	not as in Theorems C~F	not austere	12

$$(G_2^2)^* \curvearrowright (G_2 \times G_2)/G_2$$

$$(\dim (G_2 \times G_2)/G_2 = 14)$$

**Table 14.**

The positions of  $Z_0$ 's in Table 14 are as in Figure 9. The numbers  $a_5$  and  $b_3$  in Table 14 are real numbers such that  $a_4, b_2 \not\equiv \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{3\pi}{4} \pmod{\pi}$ .

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